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ABSTRACT

In this paper, we introduce the concept of τ^* -generalized pre continuous multifunctions in topological spaces and study some of their properties where τ^* is defined by $\tau^* = \{G: cl^*(G^c) = G^c\}$

KEYWORDS: τ^* -gp open set, τ^* -gp closed set, τ^* -gp continuous

1. INTRODUCTION

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have been extended to setting of multifunctions. Both functions and multifunctions are important tools for studying the properties of spaces and for constructing new spaces from previously existing ones. Many authors have introduced and studied several stronger and weaker forms of continuous functions and multifunctions. In 1970, Levine[7] introduced the concept of generalized closed sets as a generalization of closed sets in topological spaces. Using generalized closed sets, Dunham [4] introduced the concept of the closure operator Cl^* and a new topology τ^* where τ^* is defined by $\tau^* = \{G: cl^*(G^c) = G^c\}$ and studied some of their properties.

Multifunctions play a dominant role in topology and in set valued analysis, By a multifunction $F: (X, \tau) \rightarrow (Y, \sigma)$, we mean a point to set correspondence from (X, τ) into (Y, σ) with $F(x) \neq \phi$, for all $x \in X$. For a multifunction $F: (X, \tau^*) \rightarrow (Y, \sigma)$, let $F^+(B) = \{x \in X: F(x) \subseteq B\}$ and $F^-(B) = \{x \in X: F(x) \cap B \neq \emptyset\}$ where $F^+(B)$ and $F^-(B)$ denote the upper and lower inverse of a subset B of Y respectively. In particular $F^-(y) = \{x \in X: y \in F(x)\}$ for each point $y \in Y$ for each $A \subseteq X$, $F(A) = \bigcup_{x \in A} F(x)$.

In this paper, we introduce and study the concept of Generalized Pre continuous multifunctions in the Topological spaces (X, τ^*) , throughout this paper τ^* is defined by $\tau^* = \{G: cl^*(G^c) = G^c\}$

2. PRELIMINARIES

Definition: 2.1

Let A be a subset of a topological space X . Then A is called Pre-open if $A \subseteq \text{int } Cl(A)$ and Pre-closed if $Cl(\text{int}(A)) \subseteq A$. [8]

Definitions : 2.2:

A subset A of a topological space (X, τ) is called Generalized closed (briefly g-closed), if $Cl(A) \subseteq G$ whenever $A \subseteq G$ and G is open in X . [7]

Definitions : 2.3:

Strongly generalized closed (briefly strongly g-closed, if $cl(A) \subseteq G$, whenever $A \subseteq G$ and G is semi open in X . [11]

Definitions : 2.4:

Generalized pre closed (briefly gp-closed), if $\text{Clp}(A) \subseteq G$ whenever $A \subseteq G$ and G is open.[6]

Definition : 2.5:

For the subset A of a topological space X , the generalized closure operator cl^* is defined by the intersection of all g -closed sets containing A . [4]

Definition:2.6:

For the subset A of a topological space X , the topology τ^* is defined by $\tau^* = \{G: \text{cl}^*(G^c) = G^c\}$. [4]

Definition: 2.7:

A subset A of a topological space X is called τ^* -generalized closed set (briefly τ^* -gclosed). If $\text{Cl}^*(A) \subseteq G$ whenever $A \subseteq G$ and G is τ^* -open. The complement of τ^* -generalized closed set is called τ^* -generalized open (briefly τ^* -gopen). [10]

Definition: 2.8:

A subset A of a topological space X is called τ^* -generalized pre closed (briefly τ^* -gp closed). If $\text{cl}^*(\text{Clp}(A)) \subseteq G$ whenever $A \subseteq G$ and G is τ^* -open. The complement of τ^* -generalized pre closed set (briefly τ^* -gp closed) is called the τ^* -generalized pre open set (briefly τ^* -gp open). [1]

Definition: 2.9:

A function $F: X \rightarrow Y$ from a topological space X into a topological space Y is called Pre continuous if the inverse image of an open set Y is pre open in X . [8]

Definition: 2.10:

A function $F: X \rightarrow Y$ from a topological space X into a topological space Y is called g -continuous if the inverse image of a closed set in Y is g -closed in X . [3]

Definition: 2.11

A function $F: (X, \tau) \rightarrow (Y, \sigma)$ is called gp -continuous if $f^{-1}(v)$ is a gp -closed in (X, τ) for every closed set V of (Y, σ) . [2]

Definition: 2.12

A function $F: X \rightarrow Y$ from a topological space X into a topological space Y is called τ^* -gcontinuous if the inverse image of a g -closed set in Y is τ^* -gclosed in X . [5]

Definition: 2.13

A function $F: X \rightarrow Y$ from a topological space X into a topological space Y is called τ^* -gp continuous if the inverse image of every gp -open set in Y is τ^* -gopen in X .

Lemma: 2.14 [9]

For a multifunction $F: (X, \tau) \rightarrow (Y, \sigma)$, the following hold:

- (i) $G_F^+(AXB) = A \cap F^+(B)$,
- (ii) $G_F^-(AXB) = A \cap F^-(B)$, for any subsets $A \subset X$ and $B \subset Y$.

3. GENERALIZED PRE-CONTINUOUS MULTIFUNCTION

3.1 Definition

A multifunction $F: (X, \tau^*) \rightarrow (Y, \sigma)$ is said to be

- (i) Upper τ^* -generalized pre continuous if for each point $x \in X$ and open set V containing $F(x)$, there exist τ^* -generalized pre open set U of X containing x such that $F(U) \subseteq V$
- (ii) Lower τ^* -generalized pre continuous at $x \in X$ if for each open set V of Y such that $F(x) \cap V \neq \phi$, there exists τ^* -gp open set U of X containing x such that $F(U) \cap V \neq \phi$ for every $u \in U$.
- (iii) Upper (Lower) τ^* -gp continuous if F has this property at each point of X .

Theorem :3.2 For a multifunction $F: (X, \tau^*) \rightarrow (Y, \sigma)$ the following statements are equivalent

- (i) F is upper τ^* -gp continuous;
- (ii) $F^+(V) \in \tau^*$ -gpO(X) for any open set V of Y ;
- (iii) $F^-(V) \in \tau^*$ -gp closed in X for any closed set V of Y ;
- (iv) τ^* -cl* $F^-(B) \subset F^-(cl(B))$ for any $B \subset Y$
- (v) For each point $x \in X$ and each neighbourhood V of $F(x)$, $F^+(V)$ is an τ^* -generalized neighbourhood of x ;
- (vi) For each point $x \in X$ and each neighbourhood V of $F(x)$, there exists τ^* -gp neighbourhood U of x such that $F(U) \subset V$;

Proof :

(i) \Rightarrow (ii) Let V be any open set of Y and $x \in F^+(V)$. There exists, $U \in \tau^*$ -gp(X, x), such that $F(U) \subset V$. Here we obtain $x \in U$, τ^* -gpcl(U) $\subset \tau^*$ -gpcl $F^+(V)$ we have $F^+(V) \subset \tau^*$ -gpcl $F^+(V)$ and hence $F^+(V) \in \tau^*$ -gp open.

(ii) \Rightarrow (iii) Let G be any open set of X . Then $Y - G$ is closed set in Y , From (ii) $F^+(X - G) = X - F(G)$ is τ^* -gp open set in X and hence $F^-(G)$ is τ^* -gp closed set in Y .

(iii) \Rightarrow (iv) For any subset B of Y , $cl(B)$ is closed in Y and $F^-(cl(B))$ is τ^* -gp closed in Y , therefore τ^* -gpcl ($F^-(B) \subset F^-(cl(B))$).

(iv) \Rightarrow (iii) Let V be any closed set of Y . Then we have τ^* -gpcl ($F^-(V)$) $\subset F^-(cl(V)) = F^-(V)$. Hence $F^-(V)$ is τ^* -gp closed in X .

(ii) \Rightarrow (v) Let $x \in X$ and V be a neighbourhood of $F(x)$. There exist an open set G of Y such that $F(x) \subset G \subset V$, we obtain $x \in F^+(G) \in \tau^*$ -gpO(X), $F^+(G)$, $F^+(G)$ is an τ^* -gp neighbourhood of x .

(v) \Rightarrow (vi) Let $x \in X$ and V be a neighbourhood of $F(x)$. Put $U = F^+(V)$, then U is an τ^* -gp neighbourhood of x and $F(U) \subset V$.

(vi) \Rightarrow (i) Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. Then V is a neighbourhood of $F(x)$. There exist an τ^* -gp neighbourhood U of x such that $F(U) \subset V$, therefore there exist an τ^* -gpO (X) such that $x \in A \subset U$ and hence $F(A) \subset V$.

Theorem :3.3

The multifunction $F: (X, \tau^*) \rightarrow (Y, \sigma)$ is upper τ^* -gp continuous if and only if for all open set of Y , $F^+(A)$ is open in X .

Proof:

Let A be an open set of Y and $x \in F^+(A)$. Since F is upper τ^* -gp continuous. There exist a τ^* -gp open set U of x , such that $F(U) \subseteq A$, then $F^+(A)$ is open in X . Suppose $F^+(A)$ is open, Let $x \in F^+(A)$, then $F^+(A) = \{x \in A: F(x) \subseteq A\}$. Hence F is τ^* -gp upper continuous.

Theorem : 3.4

Let $X = G_1 \cup G_2$ where G_1 and G_2 are τ^* -gp closed set in X . Let $F: G_1 \rightarrow Y, F: G_2 \rightarrow Y$ be upper τ^* -gp continuous. If $F(x) = G(x)$ for each $x \in G_1 \cup G_2$. Then $H: G_1 \cup G_2 \rightarrow Y$ such that $H = \begin{cases} F(x) & \text{if } x \in G_1 \\ G(x) & \text{if } x \in G_2 \end{cases}$ is upper τ^* -gp continuous.

Proof:

Let A be an open set in Y , clearly $H^+(A) = F^+(A) \cup G^+(A)$. Since F is upper τ^* -gp continuous, F^+ is τ^* -gp open in G_1 . But G_1 is τ^* -gp open in X . Therefore $F^+(A)$ is τ^* -gp open in X . Similarly $G^+(A)$ is τ^* -gp open in G_2 and hence τ^* -gp open in X . Since union of τ^* -gp open sets is τ^* -gp open, so that $H^+(A) = F^+(A) \cup G^+(A)$ is τ^* -gp open in X . Hence H is upper τ^* -gp continuous.

Theorem: 3.5

Let $F: (X, \tau^*) \rightarrow (Y, \sigma)$ and $G: (Y, \sigma) \rightarrow (Z, \eta)$ be multifunctions. If $F: (X, \tau^*) \rightarrow (Y, \sigma)$ is upper τ^* -generalized pre continuous and $G: (Y, \sigma) \rightarrow (Z, \eta)$ is upper continuous, then $F \circ G: X \rightarrow Z$ is upper τ^* -generalized pre continuous.

Proof:

Let V be any open subset of Z using the definition of GoF , We obtain $(GoF)^+(V) = F^+(G^+(V))$. Since G is upper continuous, it follows that $G^+(V)$ is an open set. Since F is upper τ^* -generalized pre continuous, $F^+(G^+(V))$ is an τ^* -generalized pre open set. Hence GoF is upper τ^* -generalized pre continuous multifunctions.

Theorem: 3.6

Let $F: (X, \tau^*) \rightarrow (Y, \sigma)$ be a multifunction such that $F(x)$ is compact for each $x \in X$. Then F is upper τ^* -generalized pre continuous if and only if $G_F: X \rightarrow X \times Y$ is upper τ^* -generalized pre continuous.

Proof:

suppose that $F: X \rightarrow Y$ is upper τ^* -generalized pre continuous. Let $x \in X$ and H be any open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$, there exist an open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x, y) \in U(y) \times V(y) \subset H$. The family of a $\{V(y): y \in F(x)\}$ is an open cover for $F(x)$ and $F(x)$ is compact. Therefore, there exist a finite number of points, says, y_1, y_2, \dots, y_n in $F(x)$. Such that $F(x) \subset \cup\{V(y_i): 1 \leq i \leq n\}$. Set $U = \cap\{U(y_i): 1 \leq i \leq n\}$ and $V = \cup\{V(y_i): 1 \leq i \leq n\}$. Then U and V are open in X and Y , respectively and $\{x\} \times F(x) \subset U \times V \subset H$. Since F is upper τ^* -generalized pre continuous, such that $F(V_o) \subset V$. By lemma, 2.14 we have $U \cap U_o \subset U \cap F^-(V) = G_F^+(U \times V)$, Here we obtain $U \cap U_o \in \tau^*$ -gpO(X, x) and $G_F(U \cap U_o) \subset H$. This shows that G_F is upper τ^* -generalized pre continuous. Assume that $G_F: X \rightarrow X \times Y$ is upper τ^* -gp continuous. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_F(x) \subset X \times V$, there exists $U \in \tau^*$ -gpO(X, x) such that $G_F(U) \subset X \times V$. Therefore, by lemma 2.14, we have $U \subset G_F^+(X \times V) = F^+(V)$ and hence $F(U) \subset V$. This shows that F is upper τ^* -gp continuous.

Theorem: 3.7

Suppose that $(X, \tau^*), (Y, \sigma), (Z, \eta)$ are topological spaces and $F_1: X \rightarrow Y, F_2: X \rightarrow Z$ are multifunctions. Let $F_1 \times F_2: X \rightarrow Y \times Z$ be a multifunctions which is defined by $(F_1 \times F_2)(X) = F_1(X) \times F_2(X)$ for each $x \in X$. If $F_1 \times F_2$ is an upper τ^* -gp continuous multifunctions then F_1 and F_2 are upper τ^* -gp continuous multifunctions.

Proof:

Let $x \in X$ and let $K \subseteq Y$ and $H \subseteq Z$ open sets such that $x \in F^+(K)$ and $x \in F^+(H)$ and $F_1(X) \times F_2(X) = F_1 \times F_2(X) \subseteq K \times H$. We have $x \in (F_1 \times F_2)^+(K \times H)$. Since $F_1 \times F_2$ is an upper τ^* -gp continuous multifunctions.

There exist a τ^* -gp open set U containing x , such that $U \subseteq (F_1 \times F_2)^+(K \times H)$. We obtain that $U \subseteq F^+(K)$ and $U \subseteq F^+(H)$. Hence F_1 and F_2 are upper τ^* -gp continuous multifunctions.

Theorem: 3.8

Let $f : (X, \tau^*) \rightarrow (Y, \sigma)$, be upper τ^* -gp continuous and $Y \subseteq Z$. If Y is closed subset of a topological space Z then F is upper τ^* -gp continuous.

Proof :

Let K be any closed set in Z . Then $K \cap Y$ is closed in Z , it is closed in Y . Since F is upper τ^* -gp continuous. $F^+(K \cap Y)$ is τ^* -gp closed in X . But $F(x) \in Y$ for each $x \in X$, and thus $F^+ = F^+(K \cap Y)$ is τ^* -gp closed subset of X . Hence F is upper τ^* -gp continuous.

Theorem: 3.9

If $F : (X, \tau^*) \rightarrow (Y, \sigma)$ is upper τ^* -gp continuous and A is τ^* -gp closed set in X , then $F_A : A \rightarrow Y$ is upper τ^* -gp continuous.

Proof :

Let K be a closed set in Y . Since F is upper τ^* -gp continuous, and then $F^+(B)$ is τ^* -gp closed in X . If $F^+(B) \cap A = A_1$ is τ^* -gp closed in X . Here the intersection of two τ^* -gp closed is τ^* -gp closed. But $F_A(B) = A$, then A_1 is τ^* -gp closed in A . Hence F_A is upper τ^* -gp continuous.

Theorem:3.10

For a multifunction $F : (X, \tau^*) \rightarrow (Y, \sigma)$ the following statements are equivalent

- (i) F is lower τ^* -gp continuous;
- (ii) For each $x \in X$ and for each τ^* -gp open set V with $F(x) \in V \neq \emptyset$ there exist a $U \in \tau^*$ -gp $0(X, x)$ such that if $y \in U$, then $F(y) \cap V \neq \emptyset$.
- (iii) $F^-(V) \in \tau^*$ -gp $0(X)$ for any τ^* -gp open set $V \subset Y$
- (iv) $F^+(W) \in \tau^*$ -gp $C(X)$ for any τ^* -gp closed set $W \subset Y$
- (v) τ^* -gpcl [$F^+(G) \subset F^+[\tau^*$ -gpcl(G)] for every subset G of Y .

Proof:

(i) \Rightarrow (ii) Let $x \in X$ and V be the τ^* -gp open set with $F(x) \cap V \neq \emptyset$. Since F is lower τ^* -gp continuous multifunction, there exist a $U \in \tau^*$ -gp $0(X, x)$ such that if $y \in U$, $F(y) \in V \neq \emptyset$.

(i) \Rightarrow (iii) Let $x \in X$ and G be a τ^* -gp open set of Y such that $x \in F^-(G)$. By (i) there exist a $U_x \in \tau^*$ -gp $0(X, x)$ such that $U_x \subset F^-(G)$. Therefore, we have $F^-(G) = \bigcup_{x \in F^-(G)} U_x$ and hence $F^-(G) \in \tau^*$ -gp $0(X)$.

(iii) \Rightarrow (i) Let $V \in \tau^*$ -gp $0(Y)$ and $x \in F^-(V)$. By (iii) $F^-(V) \in \tau^*$ -gp $0(X)$ Taking $U = F^-(V)$ we obtain $U \subset F^-(V)$.

(iii) \Rightarrow (iv) Let us consider τ^* -gpcl(Y). Then $Y - W$ is τ^* -gp open set of Y . By (iii) $F^-(Y - W) \in \tau^*$ -gp $0(X)$ Since $F^-(Y - W) = X - F^+(W)$. Hence $F^+(W) \in \tau^*$ -gpcl(X)

(iv) \Rightarrow (iii) The proof is similar,

(iv) \Rightarrow (v) For any subset G of Y , τ^* -gpcl(G) is τ^* -gp closed in Y and then $F^+[\tau^*$ -gpcl(G)] is τ^* -gp closed in X . Hence, τ^* -gpcl [$F^+(G) \subset F^+[\tau^*$ -gpcl(G)]

(v) \Rightarrow (iv) Let W be any τ^* -gp closed set in Y . Then τ^* -gpcl [$F^+(W) \subset F^+[\tau^*$ -gpcl(W)] and hence $F^+(W)$ is a τ^* -gp closed set in X .

Theorem: 3.11

The multifunction $F: (X, \tau^*) \rightarrow (Y, \sigma)$ is Lower τ^* -gp continuous if and only if for all open set of Y , $F^-(A)$ is open in X .

Proof:

Let A be an open set of Y and $x \in F^-(A)$. Since F is lower τ^* -gp continuous. There exist a τ^* -gp open set U of x , such that $F(U) \cap A \neq \emptyset$. Every $u \in U$ then $F^-(A)$ is open in X . Suppose $F^-(A)$ is open, Let $x \in F^-(A)$, then $F^-(A) = \{x \in X: F(x) \cap A \neq \emptyset\}$. Hence F is τ^* -gp lower continuous.

Theorem: 3.12

A multifunction $F: (X, \tau^*) \rightarrow (Y, \sigma)$ is lower τ^* -gp continuous multifunction if and only if $G_F: X \rightarrow X \times Y$ is lower τ^* -gp continuous.

Proof:

Suppose that $F: (X, \tau^*) \rightarrow (Y, \sigma)$ is lower τ^* -gp continuous. Let $x \in X$ and H be any open set of $X \times Y$ such that $x \in G_F^-(H)$ since $H \cap (\{x\} \times F(x)) \neq \emptyset$ there exist $y \in F(x)$ such that $(x, y) \in H$ and hence $(x, y) \in U \times V \subseteq H$ for some open sets $U \subseteq X$ & $V \subseteq Y$. Since $F(x) \cap V \neq \emptyset$, there exist a $U_0 \in \tau^*$ -gp $O(X, x)$ such that $U_0 \subset F^-(v)$ By lemma 2.14, we have $U \cap U_0 \subset U \cap F^-(v) = G_F^-(U \times V) \subset G_F^-(H)$. Moreover, $x \in U \cap U_0 \in \tau^*$ -gp $O(X, \tau^*)$ and hence G_F are lower τ^* -gp continuous. Assume that G_F is lower τ^* -gp continuous let $x \in X$ and V be open set of Y , such that $x \in F^-(v)$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is lower τ^* -gp continuous, there exist $U \in \tau^*$ -gp $O(X, x)$ such that $U \subset G_F^-(X \times V)$. By lemma 2.14, we obtain $U \subset F^-(V)$. This shows that, F is lower τ^* -gp continuous.

Theorem: 3.13

Let $F: (X, \tau^*) \rightarrow (Y, \sigma)$ and $G: (Y, \sigma) \rightarrow (Z, \eta)$ be multifunctions. If $F: (X, \tau^*) \rightarrow (Y, \sigma)$ is lower τ^* -generalized pre continuous and $G: (Y, \sigma) \rightarrow (Z, \eta)$ is lower continuous, then $F \circ G: X \rightarrow Z$ is lower τ^* -generalized pre continuous.

Proof:

Let V be any open subset of Z . Using the definition of GoF , We obtain $(GoF)^-(V) = F^-(G^-(V))$. Since G is lower continuous, it follows that $(G^-(V))$ is an open set. Since F is lower τ^* -generalized pre continuous, $F^-(G^-(V))$ is τ^* -gp open set, and hence GoF is lower τ^* -generalized pre continuous multifunctions.

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